AN EXPLICIT FACTORISATION OF THE ZETA FUNCTIONS OF DWORK HYPERSURFACES

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Abstract. Let $\mathbb{F}_q$ be a finite field with q elements, ψ a non-zero element of $\mathbb{F}_q$, and n an integer ≥ 3 prime to q. The aim of this article is to show that the zeta function of the projective variety over $\mathbb{F}_q$ defined by $X_\psi\colon x_1^n + \cdots + x_n^n - n\psi x_1 \dots x_n = 0$ has, when n is prime and X_ψ is non singular (i.e. when $\psi^n \neq 1$), an explicit decomposition in factors coming from affine varieties of odd dimension $\leq n-4$ which are of hypergeometric type. The method we use consists in counting separately the number of points of X_ψ and of some varieties of the preceding type and then compare them. This article answers, at least when n is prime, a question asked by D. Wan in his article "Mirror Symmetry for Zeta Functions".

1. Introduction

Let n be an integer ≥ 3 and $\mathbb{F}_q$ a finite field of characteristic $p \nmid n$. We consider the family of hypersurfaces of $\mathbb{P}^{n-1}_{\mathbb{F}_q}$ defined by

$$X_\psi\colon x_1^n + \cdots + x_n^n - n\psi x_1 \dots x_n = 0, \qquad \text{(Dwork family)}$$

where $\psi \in \mathbb{F}_q$ is a non-zero parameter. We will make the assumption that X_ψ is non-singular, i.e. that $\psi^n \neq 1$. We denote by $|X_\psi(\mathbb{F}_{q^r})|$ the number of points of X_ψ over an extension $\mathbb{F}_{q^r}$ of degree r of $\mathbb{F}_q$; the zeta function of X_ψ is defined by

$$Z_{X_\psi/\mathbb{F}_q}(t) = \exp\left(\sum_{r=1}^{+\infty} |X_\psi(\mathbb{F}_{q^r})| \frac{t^r}{r}\right).$$

When $q \equiv 1 \mod n$ (see [10, Theorem 7.2 page 174]) and when n is prime (see [6, Theorem 9.5 page 179]), it is possible to show that the zeta function of X_ψ takes the form

$$Z_{X_\psi/\mathbb{F}_q}(t) = \frac{\big(Q(t,\psi)R(q^\rho t^\rho, \psi)\big)^{(-1)^{n-1}}}{(1-t)(1-qt)\dots(1-q^{n-2}t)},$$

where ρ is the order of q in $(\mathbb{Z}/n\mathbb{Z})^\times$.

In this formula, $Q(t,\psi)$ is a polynomial with integer coefficients of degree $n-1$. As proved by D. Wan (see [10, §7, Eq. (14), page 173]), this factor

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comes from the zeta function of the quotient Y_ψ of $X_\psi \otimes \mathbb{F}_{q^\rho}$ by the group $\{(\zeta_1, \dots, \zeta_n) \in \mathbb{F}_{q^\rho} \mid \zeta_i^n = 1,\ \zeta_1 \dots \zeta_n = 1\}$ (Wan calls Y_ψ a "singular mirror" of X_ψ):

$$Z_{Y_\psi/\mathbb{F}_q}(t) = \frac{Q(t,\psi)^{(-1)^{n-1}}}{(1-t)(1-qt)\dots(1-q^{n-2}t)}.$$

A simple equation of Y_ψ is $(y_1 + \dots + y_n)^n = (n\psi)^n y_1 \dots y_n$.

The factor $R(t,\psi)$ is a polynomial with integer coefficients of degree

$$\frac{(n-1)^n + (-1)^n(n-1)}{n} - (n-1)$$

whose roots have absolute values $q^{-(n-4)/2}$. We are interested in describing the factorisation of R; two approaches are possible: either predict, from a theoretical point of view, the existence of a factorisation of R, or look for explicit varieties with factors in their zeta functions appearing in R. Concerning the first approach, we refer to [8]. The second approach is raised by Wan in [10, §7, page 175] who mentions that it has been solved for $n = 3$, $n = 4$ (Dwork) and $n = 5$ (Candelas, de la Ossa, and Rodriguez Villegas); a recent article of Katz [7] also talks about the subject from a different angle[1].

The aim of this article is to handle the case where n is a prime number ≥ 5 by using only properties of Gauss sums; the fact that n is prime allows to restrict to the case $q \equiv 1 \mod n$ in view of Haessig's result [6, Theorem 9.5, page 179] that, when n is prime,

$$R(qt,\psi) = R_{X_\psi/\mathbb{F}_{q^\rho}}(q^\rho t^\rho, \psi)^{1/\rho},$$

where ρ is the order of q in $(\mathbb{Z}/n\mathbb{Z})^\times$. More precisely, if we define $N_R(q^r)$ by $R(t,\psi) = \exp\big(\sum_{r=1}^{+\infty} N_R(q^r)\frac{t^r}{r}\big)$, we will show the following result (Theorem 5.10 page 18).

Theorem. *Let n be a prime number ≥ 5 such that $q \equiv 1 \mod n$. We can write*

$$N_R(q^r) = q^{\frac{n-5}{2}} N_1(q^r) + q^{\frac{n-7}{2}} N_3(q^r) + \dots + N_{n-4}(q^r), \tag{1.1}$$

where each $N_d(q^r)$ is a sum of some $|H_{d,i}(q^r)| - (q-1)^{l-1}q^{d+1-l}$, the $H_{d,i}$ being varieties of $\mathbb{A}^{d+2}_{\mathbb{F}_q}$ of hypergeometric type of odd dimension equal to d with $1 \leq d \leq n-4$ (their equations are explicitly given in §5.3 page 16).

This equality in terms of number of points translates into a factorisation of the polynomial R in terms of the zeta function of the preceding $H_{d,i}(q^r)$.

[1] His results are in terms of traces of the Frobenius of the toric hypersurfaces $x_1 \dots x_n = \lambda y_1 \dots y_m$ over a hypergeometric sheave.

This article is organised as follows. In §2, we recall the formulas concerning Gauss and Jacobi sums we will need in the rest of the article. In §3, we compute, in terms of Gauss sums, the number of points of some varieties of hypergeometric type thanks to a method similar to the one Koblitz used in [9]. In §4, we recall the formula for the number of points of X_ψ, and in §5, we compare this formula with those from §3. Finally, in §6, we detail the cases $n = 5$ (already treated by Candelas, de la Ossa, and Rodriguez Villegas in [4]) and $n = 7$. The assumptions that n is prime and that $q \equiv 1 \mod n$ will only be used starting from §5 and §4.2 respectively.

Let us mention to finish that our method does not give a geometric link between X_ψ and the varieties of hypergeometric type we consider.

2. Gauss and Jacobi sums formulas

In all this §2, $\mathbb{F}_q$ will be a finite field with q elements.

Let $\mathbf{\Omega}$ be an algebraically closed field of characteristic zero, G a finite abelian group and $\hat{G} = \mathrm{Hom}(G, \mathbf{\Omega}^*)$ its character group. Let us recall the following orthogonality formula:

$$\frac{1}{|G|} \sum_{\varphi \in \hat{G}} \varphi(g) = \begin{cases} 1 & \text{if } g = e, \\ 0 & \text{if } g \neq e, \end{cases} \tag{2.1}$$

where e is the neutral element of G. In the following, we will use this formula when $G = \mathbb{F}_q$ or $G = \mathbb{F}_q^*$.

Let us now fix a non-trivial additive character $\varphi \colon \mathbb{F}_q \to \mathbf{\Omega}^*$.

Proposition 2.1 (Orthogonality formula)**.**

$$\frac{1}{q} \sum_{a \in \mathbb{F}_q} \varphi(ax) = \begin{cases} 1 & \textit{if } x = 0, \\ 0 & \textit{if } x \neq 0. \end{cases} \tag{2.2}$$

Proof. This results from Formula (2.1) above and the fact that every additive character is of the form $x \mapsto \varphi(ax)$ for some $a \in \mathbb{F}_q$. □

Definition 2.2 (Gauss sums)**.** If $\chi \colon \mathbb{F}_q^* \to \mathbf{\Omega}^*$ is a multiplicative character, let $G(\varphi, \chi)$ be the Gauss sum

$$G(\varphi, \chi) = \sum_{x \in \mathbb{F}_q^*} \varphi(x)\chi(x).$$

If $\mathbf{1}$ is the trivial character of $\mathbb{F}_q^*$, we have $G(\varphi, \mathbf{1}) = -1$.

Proposition 2.3 (Reflection formula)**.** *If χ is a non-trivial character of $\mathbb{F}_q^*$,*

$$G(\varphi, \chi) G(\varphi, \chi^{-1}) = \chi(-1) q. \tag{2.3}$$

Proof. Let us recall the proof of this simple property (see also [2, Theorem 1.1.4 (a), page 10]). We have

$$G(\varphi,\chi)G(\varphi,\chi^{-1}) = \sum_{x,y\in\mathbb{F}_q^*} \varphi(x+y)\chi(\tfrac{x}{y}).$$

Making the change of variable $x = yz$, we obtain

$$\begin{aligned} G(\varphi,\chi)G(\varphi,\chi^{-1}) &= \sum_{y,z\in\mathbb{F}_q^*} \varphi(y(1+z))\chi(z) \\ &= \chi(-1)(q-1) + \sum_{z\in\mathbb{F}_q^*,\ z\neq -1} \Bigg(\sum_{y\in\mathbb{F}_q^*} \varphi(y(1+z))\Bigg)\chi(z). \end{aligned}$$

We conclude by making the change of variable $y' = y(1+z)$ and by using an orthogonality formula. □

Proposition 2.4 (Multiplication formula)**.** *Let $d \geq 1$ be an integer dividing $q-1$. If η is a character of $\mathbb{F}_q^*$,*

$$\frac{G(\varphi,\eta^d)}{\prod_{\chi^d=\mathbf{1}} G(\varphi,\eta\chi)} = \frac{\eta(d)^d}{\prod_{\substack{\chi^d=\mathbf{1}\\ \chi\neq\mathbf{1}}} G(\varphi,\chi)}. \tag{2.4}$$

Proof. This seemingly simple formula does not seem to admit an elementary proof; we refer the reader to [2, Theorem 11.3.5 page 355] for additional details. □

Definition 2.5 (Jacobi sums)**.** If $(\chi_1,\dots,\chi_r)$ is a finite sequence of characters of $\mathbb{F}_q^*$, we define

$$J(\chi_1,\dots,\chi_r) = \sum_{\substack{x_1,\dots,x_r\in\mathbb{F}_q^*\\ x_1+\cdots+x_r=1}} \chi_1(x_1)\dots\chi_r(x_r).$$

Proposition 2.6 (Link with Gauss sums)**.** *If $\chi_1, \dots, \chi_r$ are characters of $\mathbb{F}_q^*$ not all trivial,*

$$J(\chi_1,\dots,\chi_r) = \begin{cases} \dfrac{1}{q}\dfrac{G(\varphi,\chi_1)\dots G(\varphi,\chi_r)}{G(\varphi,\chi_1\dots\chi_r)} & \text{if } \chi_1\dots\chi_r = \mathbf{1}, \\ \dfrac{G(\varphi,\chi_1)\dots G(\varphi,\chi_r)}{G(\varphi,\chi_1\dots\chi_r)} & \text{if } \chi_1\dots\chi_r \neq \mathbf{1}. \end{cases} \tag{2.5}$$

Proof. Let us briefly recall the proof (see also [2, Theorem 10.3.1, page 302]). The additive convolution of the functions $\chi_1, \dots, \chi_r$ is defined by

$$(\chi_1 * \cdots * \chi_r)(a) = \sum_{\substack{x_1+\cdots+x_r=a\\ x_i\in\mathbb{F}_q^*}} \chi_1(x_1)\dots\chi_r(x_r).$$

It is equal to $(\chi_1\dots\chi_r)(a)J(\chi_1\dots\chi_r)$ when $a \neq 0$. To compute the value when $a = 0$, we notice that the sum of $(\chi_1 * \cdots * \chi_r)(a)$ over $a \in \mathbb{F}_q$ is

0 since at least one of the χ_i is non trivial. Thus, $(\chi_1 * \cdots * \chi_r)(0)$ is 0 if $\chi_1 \ldots \chi_r \neq \mathbf{1}$ and is $-(q-1)J(\chi_1, \ldots, \chi_r)$ if $\chi_1 \ldots \chi_r = \mathbf{1}$. Moreover,

$$\prod_{i=1}^{r} G(\varphi, \chi_i) = \sum_{a \in \mathbb{F}_q} \varphi(a)(\chi_1 * \cdots * \chi_r)(a),$$

and so

$$\prod_{i=1}^{r} G(\varphi, \chi_i) = J(\chi_1, \ldots, \chi_r) \times \begin{cases} G(\varphi, \chi_1 \ldots \chi_n) & \text{if } \chi_1 \ldots \chi_r \neq \mathbf{1}, \\ G(\varphi, \mathbf{1}) - (q-1) & \text{if } \chi_1 \ldots \chi_r = \mathbf{1}, \end{cases}$$

which shows the result. □

Proposition 2.7 (Fourier inversion formula). *For every map $f\colon \mathbb{F}_q^* \to \mathbf{\Omega}$,*

$$\forall x \in \mathbb{F}_q^*, \quad f(x) = \frac{1}{q-1} \sum_{\eta \in \widehat{\mathbb{F}}_q^*} \Big(\sum_{y \in \mathbb{F}_q^*} f(y) \eta^{-1}(y) \Big) \eta(x). \tag{2.6}$$

Proof. It is a direct consequence of the orthogonality formulas for the characters of the abelian group $\mathbb{F}_q^*$. □

Corollary 2.8. *If $x \in \mathbb{F}_q^*$,*

$$\varphi(x) = \frac{1}{q-1} \sum_{\eta \in \widehat{\mathbb{F}}_q^*} G(\varphi, \eta^{-1}) \eta(x). \tag{2.7}$$

3. Number of points of some varieties of hypergeometric type

In all of §3, n will be an integer ≥ 2 and $\mathbb{F}_q$ a finite field with q elements.

3.1. **Computation of the number of points.** We consider here some affine varieties of hypergeometric type for which we compute the number of points by using Gauss sums and taking inspiration from Koblitz [9, §5].

Theorem 3.1. *Let $k \geq l \geq 2$ be two integers and $\lambda \in \mathbb{F}_q^*$ a parameter; we denote by $H_\lambda \subset \mathbb{A}^{k+1}$ the affine variety defined by*

$$\begin{cases} y^n = x_1^{\alpha_1} \ldots x_k^{\alpha_k} (1-x_1)^{\beta_1} \ldots (1-x_{l-1})^{\beta_{l-1}} (1 - x_l - \cdots - x_k)^{\beta_l} \\ \lambda x_1 \ldots x_l = 1 \end{cases}$$

where α_i and β_i are integers ≥ 1. The number of points of H_λ over $\mathbb{F}_q$ is

$$|H_\lambda(\mathbb{F}_q)| = (q-1)^{l-1} q^{k-l} + \sum_{\substack{\chi^n = \mathbf{1} \\ \chi \neq \mathbf{1}}} \frac{1}{q-1} \sum_{\eta} N_{\lambda, \chi, \eta}\, \eta(\lambda),$$

where

$$N_{\lambda,\chi,\eta} = \frac{1}{q^\nu} \frac{G(\varphi, \chi^{\alpha_1}\eta) \ldots G(\varphi, \chi^{\alpha_l}\eta)\, G(\varphi, \chi^{\beta_1}) \ldots G(\varphi, \chi^{\beta_l}) G(\varphi, \chi^{\alpha_{l+1}}) \ldots G(\varphi, \chi^{\alpha_k})}{G(\varphi, \chi^{\alpha_1+\beta_1}\eta) \ldots G(\varphi, \chi^{\alpha_{l-1}+\beta_{l-1}}\eta) G(\varphi, \chi^{\alpha_l + \cdots + \alpha_k + \beta_l}\eta)},$$

with ν denoting the number of trivial characters among those appearing in the denominator (namely, $\chi^{\alpha_j+\beta_j}\eta$ for $1 \leq j \leq l-1$ and $\chi^{\alpha_l+\cdots+\alpha_k+\beta_l}\eta$).

Proof. To simplify, we shall write $y^n = Q(x_1, \dots, x_k)$ for the first equation defining H_λ. We have

$$|H_\lambda(\mathbb{F}_q)| = \sum_{\substack{x\in\mathbb{F}_q^k,\ y\in\mathbb{F}_q \\ y^n=Q(x) \\ \lambda x_1\dots x_l=1}} 1 = \sum_{\substack{x\in\mathbb{F}_q^k \\ \lambda x_1\dots x_l=1}} \sum_{\substack{y\in\mathbb{F}_q \\ y^n=Q(x)}} 1,$$

with

$$|\{y \in \mathbb{F}_q \mid y^n = z\}| = \begin{cases} 1 & \text{if } z = 0, \\ 1 + \sum_{\substack{\chi^n=\mathbf{1} \\ \chi\neq\mathbf{1}}} \chi(z) & \text{if } z \neq 0, \end{cases}$$

and thus

$$\begin{aligned} |H_\lambda(\mathbb{F}_q)| &= \sum_{\substack{x\in\mathbb{F}_q^k \\ \lambda x_1\dots x_l=1 \\ Q(x)=0}} 1 + \sum_{\substack{x\in\mathbb{F}_q^k \\ \lambda x_1\dots x_l=1 \\ Q(x)\neq0}} \Big(1 + \sum_{\substack{\chi^n=\mathbf{1} \\ \chi\neq\mathbf{1}}} \chi(Q(x))\Big) \\ &= \sum_{\substack{x\in\mathbb{F}_q^k \\ \lambda x_1\dots x_l=1}} 1 + \sum_{\substack{x\in\mathbb{F}_q^k \\ \lambda x_1\dots x_l=1 \\ Q(x)\neq0}} \sum_{\substack{\chi^n=\mathbf{1} \\ \chi\neq\mathbf{1}}} \chi(Q(x)) \\ &= (q-1)^{l-1}q^{k-l} + \sum_{\substack{\chi^n=\mathbf{1} \\ \chi\neq\mathbf{1}}} \sum_{\substack{x\in\mathbb{F}_q^k \\ \lambda x_1\dots x_l=1 \\ Q(x)\neq0}} \chi(Q(x)) \\ &= (q-1)^{l-1}q^{k-l} + \sum_{\substack{\chi^n=\mathbf{1} \\ \chi\neq\mathbf{1}}} \sum_{\substack{x\in\mathbb{F}_q^k \\ Q(x)\neq0}} \chi(Q(x))\delta_{\lambda x_1\dots x_l,1}, \end{aligned}$$

where $\delta_{z,z'}$ is the Kronecker delta ($= 1$ if $z = z'$ and $= 0$ otherwise). Because

$$\forall z, z' \in \mathbb{F}_q^*, \quad \delta_{z,z'} = \frac{1}{q-1} \sum_{\eta\in\widehat{\mathbb{F}}_q^*} \eta(\tfrac{z}{z'}),$$

we may write

$$\begin{aligned} |H_\lambda(\mathbb{F}_q)| &= (q-1)^{l-1}q^{k-l} \\ &\quad + \sum_{\substack{\chi^n=\mathbf{1} \\ \chi\neq\mathbf{1}}} \frac{1}{q-1} \sum_{\eta\in\widehat{\mathbb{F}}_q^*} \Big(\sum_{\substack{x\in\mathbb{F}_q^k \\ Q(x)\neq0}} \chi(Q(x))\eta(x_1\dots x_l)\Big)\eta(\lambda). \end{aligned}$$

Let us compute $N_{\lambda,\chi,\eta} = \sum_{Q(x)\neq0} \chi(Q(x))\eta(x_1\dots x_l)$. As α_i and β_i are > 0,

$$\begin{aligned} N_{\lambda,\chi,\eta} = \sum_{\substack{(x_1,\dots,x_k)\in(\mathbb{F}_q^*)^k \\ \forall i\leq l-1,\ x_i\neq1 \\ x_l+\dots+x_k\neq1}} & (\chi^{\alpha_1}\eta)(x_1)\chi^{\beta_1}(1-x_1)\dots(\chi^{\alpha_{l-1}}\eta)(x_{l-1}) \\ & \chi^{\beta_{l-1}}(1-x_{l-1})(\chi^{\alpha_l}\eta)(x_l)\chi^{\alpha_{l+1}}(x_{l+1})\dots\chi^{\alpha_k}(x_k) \\ & \chi^{\beta_l}(1-x_l-\dots-x_k). \end{aligned}$$

We recognize a product of Jacobi sums:

$$N_{\lambda,\chi,\eta} = J(\chi^{\alpha_1}\eta, \chi^{\beta_1})\dots J(\chi^{\alpha_{l-1}}\eta, \chi^{\beta_{l-1}})J(\chi^{\alpha_l}\eta, \chi^{\alpha_{l+1}}, \dots, \chi^{\alpha_k}, \chi^{\beta_l}).$$

By using Formula (2.5) page 4, we deduce that

$$N_{\lambda,\chi,\eta} = \frac{1}{q^\nu} \frac{G(\varphi,\chi^{\alpha_1}\eta)\dots G(\varphi,\chi^{\alpha_l}\eta)G(\varphi,\chi^{\beta_1})\dots G(\varphi,\chi^{\beta_l})G(\varphi,\chi^{\alpha_{l+1}})\dots G(\varphi,\chi^{\alpha_k})}{G(\varphi,\chi^{\alpha_1+\beta_1}\eta)\dots G(\varphi,\chi^{\alpha_{l-1}+\beta_{l-1}}\eta)G(\varphi,\chi^{\alpha_l+\dots+\alpha_k+\beta_l}\eta)},$$

with ν as defined in the theorem. □

Notations. Let $N_{\lambda,\chi,\eta}$ be as in the previous theorem; we define

$$N_{\lambda,\chi} = \frac{1}{q-1}\sum_{\eta\in\widehat{\mathbb{F}}_q^*} N_{\lambda,\chi,\eta}\,\eta(\lambda) \quad \text{and} \quad N_\lambda = \sum_{\substack{\chi^n=\mathbf{1}\\ \chi\neq\mathbf{1}}} N_{\lambda,\chi,\eta}.$$

Corollary 3.2. *Assume that n is odd, that none of the elements of the sequence $(\beta_1,\dots,\beta_l,\alpha_{l+1},\dots,\alpha_k)$ are divisible by n and that, for $1\le b\le n-1$, the number of terms of the sequence $\equiv b \mod n$ is equal to the number of terms $\equiv -b \mod n$ (this implies that k is even). When these conditions are met, we say we have* complete pairing. *In this case,*

$$N_{\lambda,\chi,\eta} = q^{\frac{k}{2}-\nu}\frac{G(\varphi,\chi^{\alpha_1}\eta)\dots G(\varphi,\chi^{\alpha_l}\eta)}{G(\varphi,\chi^{\alpha_1+\beta_1}\eta)\dots G(\varphi,\chi^{\alpha_{l-1}+\beta_{l-1}}\eta)G(\varphi,\chi^{\alpha_l+\dots+\alpha_k+\beta_l}\eta)},$$

where ν is the number of trivial characters appearing in the denominator.

Proof. This is an immediate consequence of the reflection formula (2.3):

$$G(\varphi,\chi^{\beta_1})\dots G(\varphi,\chi^{\beta_l})G(\varphi,\chi^{\alpha_{l+1}})\dots G(\varphi,\chi^{\alpha_k}) = q^{\frac{k}{2}}.$$

(Let us note that, because $\chi\neq\mathbf{1}$ and because each α_i and β_j are $\not\equiv 0 \mod n$, the characters appearing are all non trivial, and so the reflection formula applies with $\chi(-1)=1$ as n is odd.) □

3.2. **Link with some hypergeometric hypersurfaces.** Assume that n is odd and that $\alpha_1+\beta_1\equiv 0 \mod n$. In that case, H_λ has the same number of points as the hypersurface of $\mathbb{A}^k$ defined by

$$y^n = x_2^{\alpha_2}\dots x_k^{\alpha_k}(1-x_2)^{\beta_2}\dots(1-x_{l-1})^{\beta_{l-1}} \cdot (1-x_l-\dots-x_k)^{\beta_{l+1}}(1-\lambda x_2\dots x_l)^{\beta_1}$$

without the points where $x_2\dots x_l=0$. We recover in this way a hypersurface of the same type as in [4, §11.1] when $n=5$ (see also example 6.1 page 19).

4. Number of points of the dwork hypersurfaces

In all this §4, n denotes an integer ≥ 3 and our aim is to compute the number of points of X_ψ and then organise it into an appropriate form to relate it to the number of points of varieties of hypergeometric type considered in §3.

To compute the number of points of X_ψ in terms of Gauss sums, it is possible to use a method close to the one A. Weil used in [11] for the diagonal

case $\psi = 0$; this is done for example in [9, Theorem 2, page 13] and [10, §3]. After recalling this computation in §4.2, we will organise the terms in the same way as Candelas, de la Ossa and Rodriguez-Villegas did for the case $n = 5$ in [3, §9] and [4, §11], namely (see Theorem 4.10):

$$|X_\psi(\mathbb{F}_q)| = 1 + q + \cdots + q^{n-2} + N_{\text{mirror}} + \sum N_s.$$

In §5, we will explain how each N_s is related to a $N_\lambda = |H_\lambda(\mathbb{F}_q)| - (q-1)^{l-1}q^{k-l}$ from §3 (here, $\lambda = \frac{1}{\psi^n}$).

4.1. **Preliminaries.** The aim of this §4.1 is to set a certain number of notations useful in what follows. The groups $\mathbb{Z}/n\mathbb{Z}$, $(\mathbb{Z}/n\mathbb{Z})^\times$ and $\mathfrak{S}_n$ act on each $(s_1, \ldots, s_n) \in (\mathbb{Z}/n\mathbb{Z})^n$ satisfying $s_1 + \cdots + s_n = 0$ in the following way:

$$\begin{aligned}
&\forall j \in \mathbb{Z}/n\mathbb{Z}, && j \cdot (s_1, \ldots, s_n) = (s_1 + j, \ldots, s_n + j);\\
&\forall k \in (\mathbb{Z}/n\mathbb{Z})^\times, && k \times (s_1, \ldots, s_n) = (ks_1, \ldots, ks_n);\\
&\forall \sigma \in \mathfrak{S}_n, && {}^\sigma(s_1, \ldots, s_n) = \big(s_{\sigma^{-1}(1)}, \ldots, s_{\sigma^{-1}(n)}\big).
\end{aligned}$$

Definition 4.1. Consider an $s = (s_1, \ldots, s_n) \in (\mathbb{Z}/n\mathbb{Z})^n$ such that $s_1 + \cdots + s_n = 0$; we denote by

a) $[s] = [s_1, \ldots, s_n]$ the class of $(s_1, \ldots, s_n)$ mod the action of $\mathbb{Z}/n\mathbb{Z}$;

b) $\langle s \rangle = \langle s_1, \ldots, s_n \rangle$ the class of $(s_1, \ldots, s_n)$ mod the simultaneous actions of $\mathbb{Z}/n\mathbb{Z}$ and $\mathfrak{S}_n$;

c) $\underline{s}$ the class of $(s_1, \ldots, s_n)$ mod the simultaneous actions of $\mathbb{Z}/n\mathbb{Z}$, $\mathfrak{S}_n$ and $(\mathbb{Z}/n\mathbb{Z})^\times$;

d) γ_s the number of permutations of $(s_1, \ldots, s_n)$.

Remarks 4.2.

a) The number γ_s only depends on $\underline{s}$, not on the choice of s.

b) If all the s_i are equal, then $\gamma_s = 1$.

c) If $\langle s \rangle = \langle 0, 1, 2, \ldots, n-1 \rangle$, then $\gamma_s = n!$ but the number of permutations of $[s]$ is $n!/n = (n-1)!$ (the $1/n$ comes from the fact that adding the same number to each coordinate amounts to a circular permutation).

The following lemma, which we will only use later (see Lemma 5.2), shows that, when n is prime, the number γ_s of permutations of $(s_1, \ldots, s_n)$ is almost always the same as the number of permutations of $[s_1, \ldots, s_n]$.

Lemma 4.3. *Assume that n is prime. If $\langle s_1, \ldots, s_n \rangle \neq \langle 0, 1, 2, \ldots, n-1 \rangle$, then γ_s is equal to the number of permutations of $[s_1, \ldots, s_n]$.*

Proof. If there exists $j \in \mathbb{Z}/n\mathbb{Z}$ non zero such that $(s_1 + j, \dots, s_n + j)$ is a permutation of $(s_1, \dots, s_n)$, then $\{s_1, \dots, s_n\}$ is a nonempty subset of $\mathbb{Z}/n\mathbb{Z}$ stable by $x \mapsto x + j$ and thus equal to $\mathbb{Z}/n\mathbb{Z}$ as n is prime. Consequently, $\langle s\rangle = \langle 0, 1, 2, \dots, n-1\rangle$. □

Remark 4.4. This proof shows that, when $\langle s\rangle \neq \langle 0, 1, \dots, n-1\rangle$, the only $j \in \mathbb{Z}/n\mathbb{Z}$ such that there exists $\sigma \in \mathfrak{S}_n$ satisfying ${}^{\sigma}s = s + j$ is $j = 0$.

4.2. **Formula for the number of points of X_ψ.** The aim of this §4.2 is to prove Theorem 4.5 below, stated in a slightly different form by Koblitz in [9, §3]. From now on, we resume using the notations and assumptions of the introduction: $\mathbb{F}_q$ is a finite field, n an integer ≥ 3 such that $q \equiv 1 \mod n$, $\psi \in \mathbb{F}_q$ is a non-zero parameter (but we don't yet suppose that $\psi^n \neq 1$) and X_ψ is the hypersurface of $\mathbb{P}^{n-1}_{\mathbb{F}_q}$ given by $x_1^n + \cdots + x_n^n - n\psi x_1 \dots x_n = 0$.

Theorem 4.5 (Koblitz)**.** *We have*

$$\begin{aligned}|X_\psi(\mathbb{F}_q)| &= 1 + q + \cdots + q^{n-2}\\ &\quad + \frac{1}{q-1}\sum_{[s]}\sum_{\eta\in\widehat{\mathbb{F}}_q^*}\frac{1}{q^\delta}\Big(\prod_{i=1}^n G(\varphi, \chi^{-s_i}\eta^{-1})\Big)G(\varphi, \eta^n)\eta\big(\tfrac{1}{(-n\psi)^n}\big),\end{aligned}$$

where $\delta = 0$ if one of the $\chi^{s_i}\eta$ is trivial and $\delta = 1$ otherwise.

Proof. For the sake of completeness, and because it would be just as long to deduce our formula from Koblitz', we will recall the proof given in [9, §3].

Let $f(x) = x_1^n + \cdots + x_n^n - n\psi x_1 \dots x_n$ and set

$$\begin{aligned}\nu_q(X_\psi) &= |\{x \in \mathbb{F}_q^n \mid f(x) = 0\}|;\\ \nu_q^*(X_\psi) &= |\{x \in (\mathbb{F}_q^*)^n \mid f(x) = 0\}|.\end{aligned}$$

As the product $x_1 \dots x_n$ is zero when one of the x_i is zero, we have $\nu_q(X_\psi) - \nu_q^*(X_\psi) = \nu_q(X_0) - \nu_q^*(X_0)$, i.e.

$$\nu_q(X_\psi) = \nu_q(X_0) + \nu_q^*(X_\psi) - \nu_q^*(X_0).$$

The computation of $\nu_q(X_0)$ is classical and goes back to A. Weil, also we will not recall it (see [11] or [2, Theorem 10.4.2, page 304]). By using Formula (2.5) to express everything in terms of Gauss sums and by doing the change of variable $\chi_i \mapsto \chi_i^{-1}$, here is what we find:

$$\nu_q(X_0) = q^{n-1} + \frac{q-1}{q}\sum_{\substack{\chi_i^n = \mathbf{1},\ \chi_i \neq \mathbf{1}\\ \chi_1\cdots\chi_n = \mathbf{1}}}\Big(\prod_{i=1}^n G(\varphi, \chi_i^{-1})\Big). \tag{4.1}$$

We now need to compute $\nu_q^*(X_\psi)$ and $\nu_q^*(X_0)$. Both computations rely on the same method, the only difference being that, when $\psi = 0$, the polynomial

$f(x)$ is a sum of n monomials instead of $n+1$ which slightly changes the result. We will only give the details for $\nu_q^*(X_\psi)$ when $\psi \neq 0$.

The orthogonality formula (2.2) page 3 for additive characters shows that

$$\begin{aligned}\nu_q^*(X_\psi) &= \frac{1}{q}\sum_{a\in\mathbb{F}_q}\sum_{x\in(\mathbb{F}_q^*)^n}\varphi(af(x))\\ &= \frac{(q-1)^n}{q} + \frac{1}{q}\sum_{a\in\mathbb{F}_q^*}\sum_{x\in(\mathbb{F}_q^*)^n}\left(\prod_{i=1}^n \varphi(ax_i^n)\right)\varphi(-n\psi a x_1\dots x_n).\end{aligned}$$

We now express each $\varphi(\dots)$ in terms of Gauss sums thanks to Formula (2.7) page 5:

$$\begin{aligned}\nu_q^*(X_\psi) = {}& \frac{(q-1)^n}{q}\\ &+ \frac{1}{q}\sum_{\eta_1,\dots,\eta_{n+1}\in\widehat{\mathbb{F}}_q^*}\left(\prod_{i=1}^{n+1} G(\varphi,\eta_i^{-1})\right)\left(\frac{1}{q-1}\sum_{a\in\mathbb{F}_q^*}(\eta_1\dots\eta_{n+1})(a)\right)\\ &\qquad \prod_{i=1}^n\left(\frac{1}{q-1}\sum_{x_i\in\mathbb{F}_q^*}(\eta_i^n\eta_{n+1})(x_i)\right)\eta_{n+1}(-n\psi).\end{aligned}$$

Using orthogonality formulas, the sums over a and the x_i are all non-zero (equal to $q-1$) if and only if

$$\begin{cases}\eta_1\dots\eta_n\eta_{n+1} = \mathbf{1}\\ \forall i\in[\![1;n]\!],\quad \eta_i^n\eta_{n+1} = \mathbf{1}\end{cases}\qquad \text{i.e.}\quad \exists\eta\in\widehat{\mathbb{F}}_q^*,\quad \begin{cases}\eta_i = \chi_i\eta\\ \chi_i^n = \mathbf{1} \text{ and } \chi_1\dots\chi_n = \mathbf{1}\\ \eta_{n+1} = \eta^{-n}\end{cases}$$

The character η defined in this way is not unique; indeed, if η' and χ_i' are also solutions of the system, there exists χ satisfying $\chi^n = \mathbf{1}$ such that $\eta' = \chi^{-1}\eta$ and $\chi_i' = \chi\chi_i$ for all i. This means that if R is a representative set of the n-uples $(\chi_1,\dots,\chi_n)$ of characters mod the $(\chi,\dots,\chi)$ satisfying $\chi_i^n = \mathbf{1}$ and $\chi_1\dots\chi_n = \mathbf{1}$ with $\chi^n = \mathbf{1}$, the map $(\chi_1,\dots,\chi_n,\eta)\mapsto(\chi_1\eta,\dots,\chi_n\eta,\eta^{-n})$ is a one-to-one map of $R\times\widehat{\mathbb{F}}_q^*$ onto the set of $(n+1)$-uples $(\eta_1,\dots,\eta_{n+1})$ satisfying the preceding conditions. From this, it results that, if χ is a multiplicative character of order n,

$$\begin{aligned}\nu_q^*(X_\psi) = {}& \frac{(q-1)^n}{q}\\ &+ \frac{1}{q}\sum_{[s]}\sum_{\eta\in\widehat{\mathbb{F}}_q^*}\left(\prod_{i=1}^n G(\varphi,\chi^{-s_i}\eta^{-1})\right)G(\varphi,\eta^n)\eta\left(\tfrac{1}{(-n\psi)^n}\right).\end{aligned}\tag{4.2}$$

This ends the computation of $\nu_q^*(X_\psi)$. By a similar method, we find

$$\nu_q^*(X_0) = \frac{(q-1)^n}{q} + \frac{q-1}{q} \sum_{\substack{\chi_i^n=\mathbf{1} \\ \chi_1\cdots\chi_n=\mathbf{1}}} \left(\prod_{i=1}^n G(\varphi,\chi_i^{-1})\right). \tag{4.3}$$

From (4.1) and (4.3), we obtain

$$\begin{aligned}\nu_q(X_0) - \nu_q^*(X_0) &= q^{n-1} - \frac{(q-1)^n}{q} - \frac{q-1}{q} \sum_{\substack{\chi_i^n=\mathbf{1} \\ \chi_1\cdots\chi_n=\mathbf{1} \\ \exists i,\ \chi_i=\mathbf{1}}} \left(\prod_{i=1}^n G(\varphi,\chi_i^{-1})\right), \\ &= q^{n-1} - \frac{(q-1)^n}{q} \\ &\quad - \frac{q-1}{q} \sum_{\substack{(\chi_1,\dots,\chi_n) \bmod \{(\chi,\dots,\chi)\} \\ \chi_i^n=\mathbf{1},\ \chi_1\cdots\chi_n=\mathbf{1} \\ \exists i,\ \chi_i=\mathbf{1}}} \sum_{\substack{\eta\in\widehat{\mathbb{F}}_q^* \\ \eta^n=\mathbf{1}}} \left(\prod_{i=1}^n G(\varphi,(\chi_i\eta)^{-1})\right),\end{aligned}$$

Writing $\chi_i = \chi^{s_i}$ where χ is, as above, a character of order n, we transform the first sum into a sum over the $[s]$ such that $\exists i,\ s_i = 0$; finally, we combine the terms of this sum with those satisfying $\eta^n = \mathbf{1}$ in Formula (4.2) above for $\nu_q^*(X_\psi)$. As $G(\varphi,\mathbf{1}) = -1$, we have, with δ as defined in the theorem,

$$\begin{aligned}\nu_q(X_\psi) &= \nu_q^*(X_\psi) + \nu_q(X_0) - \nu_q^*(X_0) \\ &= q^{n-1} + \sum_{[s]} \sum_{\eta\in\widehat{\mathbb{F}}_q^*} \frac{1}{q^\delta} \left(\prod_{i=1}^n G(\varphi,\chi^{-s_i}\eta^{-1})\right) G(\varphi,\eta^n)\eta\left(\tfrac{1}{(-n\psi)^n}\right).\end{aligned}$$

By counting the number of zeros in the projective space instead of the affine space, we obtain the announced formula. □

4.3. **Reorganisation of the terms.** We keep the assumptions and notations of §4.2 and suppose that n is odd. The aim of this §4.3 is to write the formula obtained for $|X_\psi(\mathbb{F}_q)|$ in Theorem 4.5 in terms of some coefficients $\beta_{(s_1,\dots,s_n),\chi,\eta}$ which we now define.

Definition 4.6. Let us consider $(s_1,\dots,s_n) \in (\mathbb{Z}/n\mathbb{Z})^n$ such that $s_1+\cdots+s_n = 0$. If χ is a multiplicative character of $\mathbb{F}_q^*$ of order n and if η is a character of $\mathbb{F}_q^*$, we set

$$\beta_{(s_1,\dots,s_n),\chi,\eta} = q^{\frac{n+1}{2}-z-\delta} \frac{G(\varphi,\eta)G(\varphi,\chi\eta)\dots G(\varphi,\chi^{n-1}\eta)}{G(\varphi,\chi^{s_1}\eta)\dots G(\varphi,\chi^{s_n}\eta)}, \tag{4.4}$$

where z denotes the number of trivial characters in the finite sequence $(\chi^{s_1}\eta,\dots,\chi^{s_n}\eta)$ and where $\delta = 0$ if $z \neq 0$ and $\delta = 1$ if $z = 0$ (this is the same δ as in Theorem 4.5).

Proposition 4.7. *With the above assumptions, we have*

$$\frac{1}{q^\delta}\left(\prod_{i=1}^{n} G(\varphi,\chi^{-s_i}\eta^{-1})\right)G(\varphi,\eta^n)\eta(\tfrac{1}{(-n\psi)^n}) = \beta_{(s_1,\dots,s_n),\chi,\eta}\,\eta(\tfrac{1}{\psi^n}).$$

Proof. Invoking the reflection formula (2.3), we obtain

$$\prod_{i=1}^{n} G(\varphi,\chi^{-s_i}\eta^{-1}) = q^{n-z}\,\frac{\eta(-1)^n}{G(\varphi,\chi^{s_1}\eta)\dots G(\varphi,\chi^{s_n}\eta)}\,,$$

and, using the multiplication formula (2.4), we get, as n is odd,

$$G(\varphi,\eta^n) = \frac{\eta(n)^n}{q^{\frac{n-1}{2}}}\,G(\varphi,\eta)G(\varphi,\chi\eta)\dots G(\varphi,\chi^{n-1}\eta).$$

With these two formulas, we deduce at once the result. □

The coefficients β defined above satisfy the following three compatibility relations respective to the actions of the groups $\mathbb{Z}/n\mathbb{Z}$, $\mathfrak{S}_n$ and $(\mathbb{Z}/n\mathbb{Z})^\times$.

Lemma 4.8. *With the same notations and assumptions as the preceding definition,*

$$\forall\sigma\in\mathfrak{S}_n,\qquad \beta_{(s_{\sigma(1)},\dots,s_{\sigma(n)}),\chi,\eta} = \beta_{(s_1,\dots,s_n),\chi,\eta}; \tag{4.5}$$

$$\forall j\in\mathbb{Z},\qquad \beta_{(s_1+j,\dots,s_n+j),\chi,\eta} = \beta_{(s_1,\dots,s_n),\chi,\chi^j\eta}; \tag{4.6}$$

$$\forall k\in(\mathbb{Z}/n\mathbb{Z})^\times,\qquad \beta_{(ks_1,\dots,ks_n),\chi,\eta} = \beta_{(s_1,\dots,s_n),\chi^k,\eta}. \tag{4.7}$$

Proof. Formula (4.5) results immediately from the definition of β. As for (4.6) and (4.7), we note that the product $G(\varphi,\eta)G(\varphi,\chi\eta)\dots G(\varphi,\chi^{n-1}\eta)$ in Formula (4.4) stays the same if we change η into $\chi^j\eta$ or if we change χ into χ^k with k prime to n. □

Proposition 4.9. *Under the same assumptions as above, the following quantities only depend on $\langle s\rangle$ (as well as on the choice of χ) and of $\underline{s}$ respectively and not on the choice of the representative $(s_1,\dots,s_n)$:*

$$N_{\langle s\rangle,\chi} = \frac{1}{q-1}\sum_{\eta\in\widehat{\mathbb{F}}_q^*}\beta_{(s_1,\dots,s_n),\chi,\eta}\,\eta(\tfrac{1}{\psi^n});$$

$$N_{\underline{s}} = \gamma_s\sum_{\langle s'\rangle\in\underline{s}} N_{\langle s'\rangle,\chi}.$$

Proof. For $N_{[s],\chi}$, we just use Formula (4.6) and the fact that $\eta\mapsto\chi^j\eta$ is a one-to-one map of $\widehat{\mathbb{F}}_q^*$ onto itself when $j\in\mathbb{Z}/n\mathbb{Z}$. For $N_{\underline{s}}$, we use Formula (4.7) and the fact that $\chi\mapsto\chi^k$ is a one-to-one map of $\{\chi\in\widehat{\mathbb{F}}_q^*\mid\chi^n=\mathbf{1}\}$ onto itself if $k\in(\mathbb{Z}/n\mathbb{Z})^\times$. □

We deduce the following result.

Theorem 4.10. *Under the preceding assumptions, we have*

$$|X_\psi(\mathbb{F}_q)| = 1 + q + \cdots + q^{n-2} + \sum_{\underline{s}} N_{\underline{s}}.$$

Remark 4.11. As we will see in §4.4 below, $N_{\underline{0}} = N_{\mathrm{mirror}}$ and, when X_ψ is non-singular (i.e. when $\psi^n \neq 1$), $N_{\underline{(0,1,2,\ldots,n-1)}} = 0$.

4.4. **Identification of some of the factors.** We keep the assumptions and notations of §4.3. Let us recall that Y_ψ denotes the "singular mirror" of X_ψ, as specified in the introduction, and we write $N_{\mathrm{mirror}} = |Y_\psi(\mathbb{F}_q)| - (1 + q + \cdots + q^{n-2})$.

Theorem 4.12 (Wan)**.** $N_{\underline{0}} = N_{\mathrm{mirror}}$.

Proof. See [10, §4]; note that the result is not known when $q \not\equiv 1 \mod n$, unless n is prime (see [6]). □

Let us recall that, in this §4, the only assumption we make on ψ is that $\psi \neq 0$.

Lemma 4.13. *We have*

$$N_{\langle 0,1,2,\ldots,n-1\rangle,\chi} = \begin{cases} 0 & \text{if } \psi^n \neq 1, \\ q^{\frac{n-1}{2}} & \text{if } \psi^n = 1, \end{cases}$$

and so the term $N_{\underline{(0,1,2,\ldots,n-1)}} = (n-1)!\, N_{\langle 0,1,2,\ldots,n-1\rangle,\chi}$ *does not contribute to the zeta function* $Z_{X_\psi/\mathbb{F}_q}(t)$ *when* $\psi^n \neq 1$ *and contributes as* $(1-q^{\frac{n-1}{2}}t)^{-(n-1)!}$ *when* $\psi^n = 1$.

Proof. When $\langle s_1, \ldots, s_n\rangle = \langle 0, 1, \ldots, n-1\rangle$, we have

$$G(\varphi, \chi^{s_1}\eta) \ldots G(\varphi, \chi^{s_n}\eta) = G(\varphi, \eta)G(\varphi, \chi\eta) \ldots G(\varphi, \chi^{n-1}\eta).$$

Moreover, the number z of trivial characters in the sequence $(\eta, \chi\eta, \ldots, \chi^{n-1}\eta)$ is equal to $1 - \delta$ with the notations of Definition 4.6 page 11, and thus

$$\beta_{(0,1,\ldots,n-1),\chi,\eta} = q^{\frac{n-1}{2}}.$$

Consequently,

$$N_{\langle 0,1,2,\ldots,n-2,n-1\rangle,\chi} = \frac{q^{\frac{n-1}{2}}}{q-1} \sum_{\eta \in \widehat{\mathbb{F}}_q^*} \eta(\tfrac{1}{\psi^n}),$$

and we conclude by using an orthogonality formula. □

Remark 4.14. A similar result was given by Candelas, de la Ossa and Rodriguez-Villegas when $q = p$ and $n = 5$ (see [3, §9.3]).

5. Link between the number of points

In all this §5, we will assume that the integer n is a prime ≥ 5 and that $q \equiv 1 \mod n$. We will only add the assumption that $\psi^n = 1$ in Theorem 5.10.

The aim of this section is to show (in §5.4) the Formula (1.1) of the introduction. More precisely, we shall show, in Theorem 5.7, that each $N_{\underline{s}}$ (with $\underline{s} \neq \underline{0}$)[2] appearing in Theorem 4.10 is equal, up to a multiplicative integer constant and a power of q, to a term of the form

$$N_\lambda = \sum_{\substack{\chi^n=\mathbf{1}\\ \chi\neq\mathbf{1}}} N_{\lambda,\chi} = \sum_{\substack{\chi^n=\mathbf{1}\\ \chi\neq\mathbf{1}}} \frac{1}{q-1} \sum_{\eta\in\widehat{\mathbb{F}}_q^*} N_{\lambda,\chi,\eta}\,\eta(\lambda), \tag{5.1}$$

where $\lambda = \frac{1}{\psi^n}$ and $N_{\lambda,\chi,\eta}$ is given by Corollary 3.2 page 7.

The crucial point is, starting from a given $\underline{s}$, to find the integers α_i and β_j which appear. For that, we define in §5.2 integers v_i and w_i from which we then define the integers α_i and β_j in §5.3. But before this, we start by a divisibility result useful for the main result.

5.1. A divisibility result.

The aim of this §5.1 is to show that the integer γ_s (from Definition 4.1 page 8) is divisible by

$$K_s = |\{k \in (\mathbb{Z}/n\mathbb{Z})^\times \mid [ks_1, \dots, ks_n] \text{ is a permutation of } [s_1, \dots, s_n]\}|.$$

This result is crucial in Theorem 5.7 to be sure that the quotient γ_s/K_s is an integer. Note that K_s only depends on $\underline{s}$, not on the choice of s.

Definition 5.1. Given $s \in (\mathbb{Z}/n\mathbb{Z})^n$ such that $s_1+\cdots+s_n = 0$, we consider the following subgroups of $\mathfrak{S}_n$:

$$\begin{aligned} S'_s &= \{\sigma \in \mathfrak{S}_n \mid {}^\sigma s = s\};\\ S_s &= \{\sigma \in \mathfrak{S}_n \mid [{}^\sigma s] = [s]\};\\ S_{\overline{s}} &= \{\sigma \in \mathfrak{S}_n \mid [{}^\sigma s] \in (\mathbb{Z}/n\mathbb{Z})^\times \cdot [s]\}. \end{aligned}$$

Let us note that, with these notations, $[\mathfrak{S}_n : S'_s]$ is the number γ_s of permutations of $(s_1, \dots, s_n)$ whereas $[\mathfrak{S}_n : S_s]$ is the number of permutations of $[s_1, \dots, s_n]$.

Lemma 5.2. *When $\underline{s} \neq \underline{0}$, the integer K_s divides $[\mathfrak{S}_n : S_s]$. Hence, when additionally $\langle s\rangle \neq \langle 0, 1, 2, \dots, n-1\rangle$, K_s divides $\gamma_s = [\mathfrak{S}_n : S'_s] = [\mathfrak{S}_n : S_s]$.*

[2] Let us note that there does not exist any $\underline{s} \neq \underline{0}$ when $n = 3$; this explains the assumption that $n \geq 5$.

Proof. We remark that

$$K_s = \frac{|S_{\overline{s}}|}{|S_s|} \cdot |\{k \in (\mathbb{Z}/n\mathbb{Z})^\times \mid [ks] = [s]\}|.$$

As $[s] \neq [0, \ldots, 0]$, we have $|\{k \in (\mathbb{Z}/n\mathbb{Z})^\times \mid [ks] = [s]\}| = 1$ and so

$$[\mathfrak{S}_n : S_s] = [\mathfrak{S}_n : S_{\overline{s}}] \cdot K_s.$$

When furthermore $\langle s \rangle \neq \langle 0, 1, 2, \ldots, n-1 \rangle$, we have $\gamma_s = [\mathfrak{S}_n : S_s]$ by Lemma 4.3 page 8, hence the result. $\square$

5.2. **Transformation of the β coefficients.** In order to relate $N_{\underline{s}}$ to a certain N_{1/ψ^n} in §5.3, we must first change the formula giving $\beta_{(s_1,\ldots,s_n),\chi,\eta}$.

Notations. Consider $(s_1, \ldots, s_n) \in (\mathbb{Z}/n\mathbb{Z})^n$ such that $s_1 + \cdots + s_n = 0$. For each $b \in \mathbb{Z}/n\mathbb{Z}$, define $k(b) = |\{i \mid s_i = b\}|$. We have

$$\sum_{b \in \mathbb{Z}/n\mathbb{Z}} k(b)b = 0 \quad \text{and} \quad \sum_{b \in \mathbb{Z}/n\mathbb{Z}} k(b) = n.$$

We also set $n' = |\{b \in \mathbb{Z}/n\mathbb{Z} \mid k(b) \neq 0\}|$ and $m = n - n'$.

Remarks 5.3.

a) The integer n' satisfies $1 \leq n' \leq n$ and we have $n' = 1$ if and only if $[s] = [0, \ldots, 0]$ and $n' = n$ if and only if $\langle s \rangle = \langle 0, 1, \ldots, n-1 \rangle$.

b) As n is prime, the integer n' is $\neq 2$. Indeed, if $k_1 b_1 + k_2 b_2 = 0$ with $k_1, k_2 \geq 1$ and $k_1 + k_2 = n$, then $k_1 \not\equiv 0 \mod n$ and $k_1(b_1 - b_2) = 0$, hence $b_1 = b_2$.

c) As n is odd, the integer n' is $\neq n-1$. Indeed, let $s_1, \ldots, s_{n-1}$ be distinct elements of $\mathbb{Z}/n\mathbb{Z}$ and denote by s_n the element of $\mathbb{Z}/n\mathbb{Z}$ not appearing in this sequence; as n is odd, we have $s_1 + \cdots + s_n = 0$, and so $2s_1 + \cdots + s_{n-1} = s_1 - s_n \neq 0$.

d) Thus, if $\langle s \rangle \neq \langle 0, 1, \ldots, n-1 \rangle$, then $m \geq 2$ and if, moreover, $[s] \neq [0]$, then $2 \leq m \leq n-3$.

Theorem 5.4. *With the preceding notations,*

$$\beta_{(s_1,\ldots,s_n),\chi,\eta} = q^{\frac{n-1}{2} - \nu} \frac{\prod_{b \in \mathbb{Z}/n\mathbb{Z},\ k(b)=0} G(\varphi, \chi^b \eta)}{\prod_{b \in \mathbb{Z}/n\mathbb{Z},\ k(b) \neq 0} G(\varphi, \chi^b \eta)^{k(b)-1}},$$

where $\nu = 0$ unless there exists b such that $\chi^b \eta = \mathbf{1}$ and $k(b) \neq 0$, in which case $\nu = k(b) - 1$.

Proof. From the definition of $\beta_{(s_1,\dots,s_n),\chi,\eta}$ (Definition 4.6 page 11), we have

$$\beta_{(s_1,\dots,s_n),\chi,\eta} = q^{\frac{n+1}{2}-z-\delta}\,\frac{\prod\limits_{b\in\mathbb{Z}/n\mathbb{Z}} G(\varphi,\chi^b\eta)}{\prod\limits_{b\in\mathbb{Z}/n\mathbb{Z}} G(\varphi,\chi^b\eta)^{k(b)}}$$

$$= q^{\frac{n+1}{2}-z-\delta}\,\frac{\prod\limits_{k(b)=0} G(\varphi,\chi^b\eta)}{\prod\limits_{k(b)\neq 0} G(\varphi,\chi^b\eta)^{k(b)-1}}\,.$$

We now have to show that $z+\delta = 1+\nu$. Recall that z is the number of trivial characters in the finite sequence $(\chi^{s_1}\eta,\dots,\chi^{s_n}\eta)$ and that $\delta = 0$ if $z\neq 0$ and $\delta = 1$ if $z = 0$. When $z = 0$, $\delta = 1$ and $\nu = 0$ hence $z+\delta = 1+\nu$. When $z \neq 0$, there exists a unique $b\in\mathbb{Z}/n\mathbb{Z}$ such that $\eta = \chi^{-b}$; we thus have $z = k(b)$, $\delta = 0$ and $\nu = k(b)-1$, hence $z+\delta = 1+\nu$. $\square$

Remark 5.5. Let $(v_1,\dots,v_m)$ be an enumeration of the $b\in\mathbb{Z}/n\mathbb{Z}$ such that $k(b) = 0$ and let $(w_1,\dots,w_m)$ be an enumeration of the $b\in\mathbb{Z}/n\mathbb{Z}$ such that $k(b)\geq 2$, each repeated with multiplicity $k(b)-1$. The formula of Theorem 5.4 can be rewritten as

$$\beta_{(s_1,\dots,s_n),\chi,\eta} = q^{\frac{n-1}{2}-\nu}\,\frac{G(\varphi,\chi^{v_1}\eta)\dots G(\varphi,\chi^{v_m}\eta)}{G(\varphi,\chi^{w_1}\eta)\dots G(\varphi,\chi^{w_m}\eta)}\,, \tag{5.2}$$

where ν is the number of trivial characters appearing in the denominator.

Lemma 5.6. *With the notations of the preceding remark,*

$$v_1+\dots+v_m \equiv w_1+\dots+w_m \mod n.$$

Proof. This identity can be rewritten as

$$\sum_{k(b)=0} b = \sum_{k(b)\geq 1} (k(b)-1)b \quad \text{i.e.} \quad \sum_b b = \sum_b k(b)b.$$

We conclude by noting that $\sum_{b\in\mathbb{Z}/n\mathbb{Z}} k(b)b = 0$ and that, because n is odd, $\sum_{b\in\mathbb{Z}/n\mathbb{Z}} b = 0$. $\square$

5.3. **Link with the hypergeometric varieties.** We now establish the link between X_ψ and the varieties of hypergeometric type from §3.

Theorem 5.7. *Let $\underline{s}$ be distinct from the class of $(0,1,\dots,n-1)$ and of $(0,\dots,0)$. If s is a representative of $\underline{s}$, assume that there exists two sequences $(v_1,\dots,v_m)$ and $(w_1,\dots,w_m)$ of elements of $\mathbb{Z}/n\mathbb{Z}$ as in Remark 5.5 and an even integer $m'\leq m-2$ such that*

$$\forall i\in [\![1;\tfrac{m'}{2}]\!],\quad w_{2i-1}-v_{2i-1} \equiv -(w_{2i}-v_{2i}) \mod n.$$

We consider the affine variety H_{1/ψ^n} of dimension $2m - m' - 3$ given by

$$\begin{cases} y^n = x_1^{v_1} \dots x_m^{v_m} x_{m+1}^{v_{m'+1} - w_{m'+1}} \dots x_{2m-m'-2}^{v_{m-2} - w_{m-2}} (1 - x_1)^{w_1 - v_1} \dots \\ \qquad\qquad (1 - x_{m-1})^{w_{m-1} - v_{m-1}} (1 - x_m - \dots - x_{2m-m'-2})^{v_{m-1} - w_{m-1}} \\ x_1 \dots x_m = \psi^n \end{cases}$$

(In this formula, we replace the exponents by their representatives in $[\![1; n]\!]$.) It is a variety of the form considered in Corollary 3.2 page 7 and we have, using the notations of §3,

$$N_{\underline{s}} = \frac{\gamma_s}{K_s} q^{\frac{n+1}{2} - \frac{2m-m'}{2}} N_{1/\psi^n} \qquad \textit{where } \gamma_s/K_s \in \mathbb{N} \textit{ by Lemma 5.2.}$$

Proof. As $\underline{s}$ is distinct from the class of $(0, 1, \dots, n-1)$, we have $m \geq 2$ (see Remark 5.3.d page 15). The variety we consider is the one introduced in Theorem 3.1 page 5 with $l = m$, $k = 2m - m' - 2$ and

$$\begin{aligned} &\alpha_1 = v_1, \quad \dots, \quad \alpha_m = v_m; \\ &\alpha_{m+1} = v_{m'+1} - w_{m'+1}, \quad \dots, \quad \alpha_{2m-m'-2} = v_{m-2} - w_{m-2}; \\ &\beta_1 = w_1 - v_1, \quad \dots, \quad \beta_{m-1} = w_{m-1} - v_{m-1}, \quad \beta_m = v_{m-1} - w_{m-1}. \end{aligned}$$

According to the pairing assumption on the v_i and w_i and to Lemma 5.6, we have

$$v_{m'+1} + \dots + v_m = w_{m'+1} + \dots + w_m \quad \text{in } \mathbb{Z}/n\mathbb{Z},$$

and thus, $\alpha_m + \alpha_{m+1} + \dots + \alpha_{2m-m'-2} + \beta_m \equiv w_m \mod n$. Moreover,

$$\begin{aligned} &\alpha_1 + \beta_1 \equiv w_1 \mod n, \quad \dots, \quad \alpha_{m-1} + \beta_{m-1} \equiv w_{m-1} \mod n; \\ &\beta_1 + \beta_2 \equiv 0 \mod n, \quad \dots, \quad \beta_{m'-1} + \beta_{m'} \equiv 0 \mod n; \\ &\alpha_{m+1} + \beta_{m'+1} \equiv 0 \mod n, \quad \dots, \quad \alpha_{2m-m'-2} + \beta_{m-2} \equiv 0 \mod n; \\ &\beta_{m-1} + \beta_m \equiv 0 \mod n. \end{aligned}$$

The last three lines show that we have complete pairing (in the sense of Corollary 3.2 page 7) of the sequence $(\beta_1, \dots, \beta_m, \alpha_{m+1}, \dots, \alpha_{2m-m'-2})$; these elements are $\not\equiv 0 \mod n$ as $v_i \not\equiv w_i \mod n$, and so

$$N_{1/\psi^n, \chi, \eta} = q^{\frac{2m-m'-2}{2} - \nu} \frac{G(\varphi, \chi^{v_1}\eta) \dots G(\varphi, \chi^{v_m}\eta)}{G(\varphi, \chi^{w_1}\eta) \dots G(\varphi, \chi^{w_m}\eta)}.$$

Hence, by comparing with Formula (5.2) page 16,

$$\beta_{(s_1, \dots, s_n), \chi, \eta} = q^{\frac{n+1}{2} - \frac{2m-m'}{2}} N_{1/\psi^n, \chi, \eta}.$$

Multiplying this equality by $\frac{1}{q-1}\eta(\frac{1}{\psi^n})$ and summing over $\eta \in \widehat{\mathbb{F}}_q^*$, we get

$$N_{\langle s \rangle, \chi} = q^{\frac{n+1}{2} - \frac{2m-m'}{2}} N_{1/\psi^n, \chi}.$$

We now sum over $k \in [\![1; n-1]\!]$ the preceding formula where χ is remplaced by χ^k. Noting that $N_{\langle s\rangle,\chi^k} = N_{\langle ks\rangle,\chi}$ (see Formula (4.7) page 12), we obtain

$$\sum_{k=1}^{n-1} N_{\langle ks\rangle,\chi} = q^{\frac{n+1}{2}-\frac{2m-m'}{2}} N_{1/\psi^n}.$$

The left hand side is equal to $K_s \sum_{\langle s'\rangle \in \underline{s}} N_{\langle s'\rangle,\chi}$ i.e. to $\frac{K_s}{\gamma_s} N_{\underline{s}}$. As $[s] \neq [0]$, Lemma 5.2 page 14 shows that γ_s/K_s is an integer. The result is hence proved. □

Remark 5.8. When $m' = m-2$, we have $v_{m-1} - w_{m-1} = w_m - v_m$ by Lemma 5.6 page 16 and the equation of the variety simplifies greatly:

$$H_{1/\psi^n} : \begin{cases} y^n = x_1^{v_1} \dots x_m^{v_m} (1-x_1)^{w_1 - v_1} \dots (1-x_m)^{w_m - v_m} \\ x_1 \dots x_m = \psi^n \end{cases}$$

5.4. **Conclusion.** We are now capable of showing Formula (1.1) of the introduction. We begin by a result giving a lower bound on the number of pairings which will enable us to show that the dimension of the hypergeometric varieties is always $\leq n-4$.

Proposition 5.9. *Let $\underline{s}$ be distinct from the class of $(0, 1, \dots, n-1)$ and of $(0, \dots, 0)$ and let s be a representative of $\underline{s}$. We can choose two sequences $(v_1, \dots, v_m)$ and $(w_1, \dots, w_m)$ satisfying the assumptions of Remark 5.5 page 16 such that we have the pairing*

$$\forall i \in [\![1; \tfrac{2m-n+1}{2}]\!], \quad w_{2i-1} - v_{2i-1} \equiv -(w_{2i} - v_{2i}) \mod n.$$

Proof. Let $(v_1, \dots, v_m)$ and $(w_1, \dots, w_m)$ be sequences as in Remark 5.5. By Theorem 1.2 of [1, page 126], it is possible to permute $(w_1, \dots, w_m)$ so that the $v_i - w_i$ are pairwise distinct. Define V as the subset $\{v_i - w_i\}$ of $\mathbb{Z}/n\mathbb{Z}$ (it has m elements) and μ as the number of opposite pairs contained in V; we have

$$2\mu = |V \cap (-V)| = 2m - |V \cup (-V)| \geq 2m - (n-1).$$

As 2μ is the maximal number of pairings, this ends the proof. □

Theorem 5.10. *If $\psi^n \neq 1$, we can write*

$$\begin{aligned} |X_\psi(\mathbb{F}_q)| = 1 + q + \dots + q^{n-2} + N_{\text{mirror}} \\ + q^{\frac{n-3}{2}} N_1 + q^{\frac{n-5}{2}} N_3 + \dots + q N_{n-4}, \end{aligned}$$

where each N_d is a sum of terms of the form $|H_\lambda(\mathbb{F}_q)| - (q-1)^{l-1} q^{d+1-l}$ where α_i and β_j are obtained from each $\underline{s}$ as described in §5.3 and where each $H_\lambda \subset \mathbb{A}^{d+2}$ is a variety of hypergeometric type of odd dimension equal to d with $1 \leq d \leq n-4$ (here, $\lambda = 1/\psi^n$) as considered in §3.1.

Proof. We saw in Theorem 4.10 page 13 that, if $\psi \neq 0$ and $q \equiv 1 \mod n$, we could write

$$|X_\psi(\mathbb{F}_q)| = 1 + q + \cdots + q^{n-2} + N_{\underline{0}} + \sum_{\underline{s} \neq \underline{0}} N_{\underline{s}}.$$

In Theorem 4.12, we recalled Wan's result showing that $N_{\underline{0}} = N_{\text{mirror}}$ and in Lemma 4.13, we showed that the term corresponding to $(0, 1, 2, \ldots, n-1)$ was zero when $\psi^n \neq 1$.

Let us now consider $\underline{s}$ distinct from the class of $(0, \ldots, 0)$ and of $(0, 1, 2, \ldots, n-1)$. Let m' be the greatest even integer $\leq m-2$ such that there exists two sequences $(v_1, \ldots, v_m)$ and $(w_1, \ldots, w_m)$ as in Remark 5.5 verifying

$$\forall i \in [\![1; \tfrac{m'}{2}]\!], \quad w_{2i-1} - v_{2i-1} \equiv -(w_{2i} - v_{2i}) \mod n.$$

By Proposition 5.9, we have $m' \geq 2m - n + 1$ (note that, by Remark 5.3.d page 15, $m + 3 \leq n$, hence $m - 2 \geq 2m - n + 1$). The dimension $d = 2m - m' - 3$ of the corresponding variety of hypergeometric type considered in Theorem 5.7 page 16 thus satisfies $1 \leq d \leq n - 4$.

Moreover, we have $q^{\frac{n+1}{2} - \frac{2m-m'}{2}} = q^{\frac{n-d-2}{2}}$, and so, as d varies between 1 and $n-4$, these powers of q take the values $q^{\frac{n-3}{2}}, \ldots, q$ respectively and all these values are obtained; indeed, if we consider an integer m such that $2 \leq m = d+1 \leq n-3$ and define $s = (0, \ldots, 0, 1, n-1, 2, n-2, \ldots, \frac{n-m-1}{2}, n - \frac{n-m-1}{2})$, then $w = (0, \ldots, 0)$ and $v = (\frac{n-m+1}{2}, n - \frac{n-m+1}{2}, \ldots, \frac{n-1}{2}, \frac{n+1}{2})$ each consist of m elements and we have $m' = m - 2$ with the notations of Theorem 5.7. □

6. Examples

To illustrate the methods we have just presented in this paper, let's detail explicitly the cases $n = 5$ and $n = 7$; these examples are given in terms of the hypersurfaces of hypergeometric type from §3.2.

Example 6.1 ($n = 5$)**.** Let's recover the results announced by Candelas, de la Ossa and Rodriguez-Villegas in [4] in the non-singular and non-diagonal case (see [5] for a complete treatment of the $n = 5$ case). We are interested in the factorisation of the zeta function of the quintic $\mathcal{M}_\psi \colon x_1^5 + \cdots + x_5^5 - 5\psi x_1 \ldots x_5 = 0$ when $\psi \neq 0$ and $\psi^5 \neq 1$. We list all the classes $\underline{(s_1, \ldots, s_5)} \neq \underline{(0, 0, 0, 0, 0)}$ and $\neq \underline{(0, 1, 2, 3, 4)}$ (following the notations from §§4.1, 5.1, 5.2 and 5.3):

$\underline{s}$	γ_s	K_s	m	m'	d
$(0,0,0,1,4)$	20	2	2	0	1
$(0,0,1,1,3)$	30	2	2	0	1

Using the method described above, we obtain the following table (the hypergeometric hypersurfaces are all of the form $y^5 = x^{v_1}(1-x)^{v_2}(1-\frac{1}{\psi^5}x)^{5-v_2}$).

$\underline{s}$	v_1	v_2	w_1	w_2	EQUATION
$(0,0,0,1,4)$	2	3	0	0	$y^5 = x^2(1-x)^3(1-\frac{1}{\psi^5}x)^2$
$(0,0,1,1,3)$	2	4	0	1	$y^5 = x^2(1-x)^4(1-\frac{1}{\psi^5}x)$

We find the same equations as those given in [4, §11.1]:

$$\mathcal{A}_\psi \colon y^5 = x^2(1-x)^3(1-\tfrac{1}{\psi^5}x)^2 \quad \text{and} \quad \mathcal{B}_\psi \colon y^5 = x^2(1-x)^4(1-\tfrac{1}{\psi^5}x).$$

We set $N_{\mathcal{A}_\psi} = |\mathcal{A}_\psi(\mathbb{F}_q)| - q$ and $N_{\mathcal{B}_\psi} = |\mathcal{B}_\psi(\mathbb{F}_q)| - q$ (these number of points are affine). We have, when $\psi \neq 0$, $\psi^5 \neq 1$ and $q \equiv 1 \mod 5$:

$$|\mathcal{M}_\psi(\mathbb{F}_q)| = 1 + q + q^2 + q^3 + N_{\text{mirror}} + 10qN_{\mathcal{A}_\psi} + 15qN_{\mathcal{B}_\psi}.$$

Example 6.2 ($n = 7$)**.** We use the preceding results to find the factorisation of the zeta function of the septic $S_\psi \colon x_1^7 + \cdots + x_7^7 - 7\psi x_1 \ldots x_7 = 0$. We list the $\underline{(s_1, \ldots, s_7)} \neq \underline{(0, \ldots, 0)}$ and $\neq \underline{(0,1,2,3,4,5,6)}$ (following the notations from §§4.1, 5.1, 5.2 and 5.3):

$\underline{s}$	γ_s	K_s	m	m'	d
$(0,0,0,1,2,5,6)$	840	2	2	0	1
$(0,0,1,1,3,4,5)$	1260	2	2	0	1
$(0,0,1,1,2,4,6)$	1260	2	2	0	1
$(0,0,0,0,1,2,4)$	210	3	3	0	3
$(0,0,0,1,1,2,3)$	420	1	3	0	3
$(0,0,1,1,3,3,6)$	630	3	3	0	3
$(0,0,0,0,0,1,6)$	42	2	4	2	3
$(0,0,0,0,1,1,5)$	105	1	4	2	3
$(0,0,0,1,1,1,4)$	140	2	4	2	3
$(0,0,0,1,1,6,6)$	210	2	4	2	3

The result is that, when $\psi \neq 0$, $\psi^7 \neq 1$ and $q \equiv 1 \mod 7$, the number of points takes the form

$$|S_\psi(\mathbb{F}_q)| = 1 + q + q^2 + q^3 + q^4 + q^5 + N_{\text{mirror}} + q^2 N_1 + qN_3,$$

where the terms corresponding to curves of $\mathbb{A}^2$ can be written as

$$N_1 = 420N_{c_1} + 630N_{c_2} + 630N_{c_3},$$

and those corresponding to threefold hypersurfaces of $\mathbb{A}^4$ can be written as

$$N_3 = 70N_{t_1} + 420N_{t_2} + 210N_{t_3} + 21N_{t'_1} + 105N_{t'_2} + 70N_{t'_3} + 105N_{t'_4},$$

where the various terms are defined in the following table (the corresponding number of points are in the affine space).

EQUATION OF THE HYPERSURFACE	NB. OF PTS.
$y^7 = x^3(1-x)^4(1-\frac{1}{\psi^7}x)^3$	$q + N_{c_1}$
$y^7 = x^2(1-x)^6(1-\frac{1}{\psi^7}x)$	$q + N_{c_2}$
$y^7 = x^3(1-x)^5(1-\frac{1}{\psi^7}x)^2$	$q + N_{c_3}$
$y^7 = x_1^3x_2^5x_3^3(1-x_1)^4(1-x_2-x_3)^6(1-\frac{1}{\psi^7}x_1x_2)$	$q^3 + N_{t_1}$
$y^7 = x_1^4x_2^5x_3^4(1-x_1)^3(1-x_2-x_3)^6(1-\frac{1}{\psi^7}x_1x_2)$	$q^3 + N_{t_2}$
$y^7 = x_1^2x_2^4x_3^4(1-x_1)^6(1-x_2-x_3)^5(1-\frac{1}{\psi^7}x_1x_2)^2$	$q^3 + N_{t_3}$
$y^7 = x_1^2x_2^5x_3^3(1-x_1)^5(1-x_2)^2(1-x_3)^4(1-\frac{1}{\psi^7}x_1x_2x_3)^3$	$q^3 + N_{t'_1}$
$y^7 = x_1^3x_2^3x_3^2(1-x_1)^4(1-x_2)^4(1-x_3)^6(1-\frac{1}{\psi^7}x_1x_2x_3)$	$q^3 + N_{t'_2}$
$y^7 = x_1^3x_2^5x_3^2(1-x_1)^4(1-x_2)^3(1-x_3)^6(1-\frac{1}{\psi^7}x_1x_2x_3)$	$q^3 + N_{t'_3}$
$y^7 = x_1^3x_2^5x_3^2(1-x_1)^4(1-x_2)^3(1-x_3)^4(1-\frac{1}{\psi^7}x_1x_2x_3)^3$	$q^3 + N_{t'_4}$

Let's justify for example the equation corresponding to $[0,0,0,0,0,1,6]$. We have $\{v_1, v_2, v_3, v_4\} = \{2,3,4,5\}$ and $w_1 = w_2 = w_3 = w_4 = 0$. Let's take, for example, $v_1 = 2$, $v_2 = 5$, $v_3 = 3$ and $v_4 = 4$ so that $w_1 - v_1 = -(w_2 - v_2)$ and $w_3 - v_3 = -(w_4 - v_4)$. For this choice, we have $m = 4$, $m' = m - 2 = 2$ and the equation we obtain is

$$y^7 = x_1^2x_2^5x_3^3(1-x_1)^5(1-x_2)^2(1-x_3)^4(1-\tfrac{1}{\psi^7}x_1x_2x_3)^3.$$

This is the equation corresponding to $N_{t'_1}$. The other equations follow in a similar way.

Remark 6.3. Using the same method, we could treat the cases $n = 11$, $n = 13$, etc. The only difficulty is practical, as the number of classes $\underline{(s_1, \ldots, s_n)}$ grows quickly with n.

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